J. Fluid Mech. (1974), vol. 66, part 2, pp. 241–260
Printed in Great Britain

On Laplace's tidal equations

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(Received 29 November 1973)

The parametric limit process for Laplace's tidal equations (LTE) is considered, starting from the full equations of motion for a rotating, gravitationally stratified, compressible fluid. The boundary-value problem for free oscillations of angular frequency σ is not well posed if $\sigma^2 < N^2 + 4\omega^2$, where N is the Väisälä frequency and ω is the rotational speed of the Earth, and the governing partial differential equation is elliptic/hyperbolic on the polar/equatorial sides of the inertial latitudes given by $\pm\sigma = f$ (vertical component of $2\boldsymbol{\omega}$) if $\sigma < 2\omega \ll N$. The solution of this ill-posed problem is considered for a global ocean of uniform depth, with the effects of ellipticity, the 'traditional' approximation and stratification measured by the small parameters $m = \omega^2 a/g$, $\delta = h/a$ and $s = hN^2/g$ (g = acceleration due to gravity, h = depth of ocean, a = radius of Earth). LTE represent the joint limit $m, \delta, s \downarrow 0$ and yield bounded solutions for all latitudes. It is argued that the parametric expansion in m is regular. The joint expansion in δ and s with LTE as the basic approximation is singular at the inertial latitudes if $\sigma < 2\omega$, which difficulty is traced to the failure of LTE to provide an adequate description of the characteristics in the hyperbolic domain. It is shown that an alternative formulation, in which the buoyancy force is retained in the basic equations in the joint limit $s \downarrow 0$, $\delta \downarrow 0$ with $N \gg 2\omega$, yields solutions that are uniformly valid in the neighbourhoods of the inertial latitudes. The resulting representation comprises a barotropic mode, which satisfies LTE, and an infinite discrete set of baroclinic modes, each of which has Airy turning points at the inertial latitudes and is trapped between them. The barotropic and baroclinic modes are coupled by the Coriolis acceleration associated with the horizontal component of the Earth's rotation. The relative effects of this coupling are uniformly $O(\delta)$ if $\sigma > 2\omega$, but it induces currents $O(\delta/s^{\frac{1}{2}})$ and vertical displacements $O(\delta/s^{\frac{3}{4}})$ between the inertial latitudes if $\sigma < 2\omega \ll N$. It appears that resonant amplification of the baroclinic modes forced by the barotropic modes could imply internal displacements that dominate those of the basic motion.

1. Introduction

Laplace's tidal equations (Lamb 1932, §215; Eckart 1960, chap. 7), hereinafter LTE, have formed the basis of the dynamical theory of the tides for almost two centuries; nevertheless, despite many controversies [see Hendershott & Munk (1970) for a recent review], the parametric limit process implicit in their derivation does not appear to have been critically examined. A definitive investigation of this process is inhibited by the difficulty of solving even LTE without further approximations, and Laplace's global ocean of uniform depth (Longuet-Higgins 1968) and the corresponding hemispherical ocean (Longuet-Higgins & Pond 1970) are the only configurations for which essentially complete solutions are available. It is reasonable to assume, however, that continental boundaries and submarine topography do not alter the qualitative character of the basic perturbation process, and the following investigation therefore adopts the model of a global ocean.

Appropriate scaling parameters for the description of such an ocean, together with the approximate numerical values obtained by distributing the real oceans uniformly over the Earth, are

$$\delta = h/a \doteq 6\times 10^{-4}, \tag{1.1}$$

$$\epsilon \doteq \mathscr{m} = \omega^2 a/g \doteq 3\times 10^{-3}, \tag{1.2}$$

$$\Gamma = gh/c^2 \doteq 2\times 10^{-2} \tag{1.3}$$

and

$$s = -(h\rho'/\rho) - \Gamma \equiv hN^2/g \doteq 2\times 10^{-3}, \tag{1.4}$$

where $h \doteq 4\times 10^3$ m is the depth; $a = 6{\cdot}4\times 10^6$ m is the radius of the Earth; ϵ is the ellipticity of the Earth ($1-\epsilon$ is the ratio of the polar radius to the equatorial radius); $\mathscr{m}$ is the kinematic ellipticity; $\omega \doteq 7{\cdot}3\times 10^{-5}$ rad/s is the angular velocity of the Earth; $g \doteq 10$ m/s^2; Γ and s are measures of compressibility and stratification; $c \doteq 1{\cdot}4\times 10^3$ m/s is the velocity of sound; ρ and ρ' are the density and its vertical gradient; $N \doteq 2\times 10^{-3}$ rad/s is a representative buoyancy (Väisälä) frequency (N varies by at least an order of magnitude between surface and bottom). Derived similarity parameters are

$$\beta = 4\omega^2 a^2/gh \equiv 4\mathscr{m}/\delta \doteq 20, \tag{1.5}$$

$$\mathcal{N} = N/2\omega \equiv (s/4\mathscr{m}\delta)^{\frac{1}{2}} \doteq 10 \tag{1.6}$$

and

$$\lambda = \sigma/2\omega < 1, \tag{1.7}$$

where σ is the angular frequency.

The basic approximations and idealizations for LTE (in the order in which they are typically invoked in a derivation from the basic equations of motion) are

(i) a perfect homogeneous fluid ($\Gamma = s = 0$),

(ii) small disturbances relative to a state of uniform rotation,

(iii) a spherical Earth ($\epsilon = \mathscr{m} = 0$),

(iv) a uniform gravitational field,

(v) a rigid ocean bottom,

(vi) a shallow ocean ($\delta \ll 1$), in which both the Coriolis acceleration associated with the horizontal component of the Earth's rotation and the vertical component of the particle acceleration are neglected.

In addition, it is usually assumed that the motion is simple harmonic; however, this assumption is (in principle) no more restrictive than (ii) by virtue of Fourier's theorem.

(i) The assumption of a perfect fluid is amply justified in the present context and is not considered further (turbulent, in contrast to molecular, diffusion may be important in the description of resonant oscillations but must be treated empirically). The assumption of a homogeneous fluid (Γ, $s \downarrow 0$) implies barotropic motion and rules out the internal waves (baroclinic motion) that are observed in the real oceans. The vertical displacement scale for the latter is much greater than that for the barotropic motion, and it therefore may be necessary to allow for the buoyancy force implied by stratification even though $s \ll 1$. Compressibility, on the other hand, is relatively unimportant (which is to say that sound waves are unimportant compared with gravity waves) and may be neglected except as it enters s (Boussinesq approximation); however, $s \ll \Gamma$ for approximately adiabatic stratification, as in the oceans outside the thermocline, and a rational perturbation analysis therefore must consider the limit $\Gamma \downarrow 0$. [We note that $\Gamma \doteqdot 1$ for the atmosphere and that this, together with the absence of a free surface, prevents a sharp distinction between barotropic and baroclinic motion (cf. Chapman & Lindzen 1970, chap. 3).]

(ii) The assumption of small disturbances, which permits linearization of the equations of motion, appears to be amply justified for tidal motion except in very shallow water and requires no further discussion in the present context.

(iii) A rational appraisal of the limit $m \downarrow 0$ must recognize that m is significantly larger than δ. However, the effects of ellipticity are essentially geometrical, and neglecting them implies an error that is uniformly $O(m)$. These effects could be included by referring the limit $\delta \downarrow 0$ to spheroidal, rather than spherical, level surfaces, but this would complicate the mathematical analysis without contributing significantly to the end results, for which a *uniform* error $O(m)$ is negligible. [Solberg (1936*a*), who, according to Proudman (1942), "considered it an 'internal contradiction' to retain the dynamical effect of the Earth's rotation on the tides and yet to neglect it on the shape of the mean surface of the ocean", gives a lengthy treatment of tidal oscillations on a spheroidal Earth. As Proudman remarks, the 'contradiction' is resolved by regarding m and λ as independent parameters. It also should be remarked that current satellite data on the figure of the Earth imply that significant departures from a spheroid occur at $O(\epsilon^2)$, so that there is no geophysical justification in going beyond terms $O(\epsilon)$.]

(iv, v) The assumption of a uniform gravitational field implies the neglect of both the radial variation of gravity (the neglect of the tangential variation of gravity is implicit in the limit $m \downarrow 0$) and self-attraction (mutual gravitational attraction associated with free-surface displacement). The radial variation of gravity is $O(\delta)$ and therefore should be included in a completely consistent investigation of the limit $\delta \downarrow 0$; however, the effects of this variation are found (by analysis) to be uniformly $O(\delta)$ and quantitatively less (or at least no more) significant than those of ellipticity. Self-attraction is quantitatively significant for the tides and may be formally incorporated in LTE either through the introduction of a linear operator (cf. Platzman 1971) or through the expansion of the solution in spherical harmonics (Lamb 1932, § 200). A similar statement holds for the effects of bottom elasticity. However, including either self-attraction or bottom elasticity in the following perturbation analysis would introduce additional complications in an already complicated analysis, and we therefore fall back on the intuitive argument that neither is qualitatively significant in the present context.

(vi) The assumptions grouped in (vi) above are designated collectively as the *hydrostatic* or *traditional* (Eckart 1960, p. 95ff) approximation. This approximation, which corresponds to the limit $\delta \downarrow 0$ with β and λ fixed, has long been controversial (see Bjerknes *et al.* 1933, chap. 13; Solberg 1936*a*; Proudman 1942; Phillips 1966, 1968; Veronis 1968), but the crucial fact that the limit $\delta \downarrow 0$ may be singular for $\lambda < 1$ appears to have been overlooked prior to Stewartson & Rickard's (1969) study of free oscillations of a rotating, homogeneous, inviscid fluid between concentric spheres (corresponding to the present problem for $\beta = \mathcal{N} = 0$).

Stewartson & Rickard posed a perturbation expansion in powers of δ and discovered that terms $O(\delta^n)$, $n \geq 1$, in this expansion are singular at those two latitudes (hereinafter *inertial latitudes*) at which

$$f \equiv 2\omega\mu = \pm\sigma, \tag{1.8}$$

or, equivalently, $\mu = \pm\lambda$, where μ is the sine of the latitude. They then attempted to construct an inner expansion in the stretched co-ordinate $\delta^{-\frac{1}{2}}(\mu-\lambda)$ but were unable to match their inner and outer expansions and concluded that the perturbation solution was 'pathological'. Stewartson & Walton (1974, private communication) subsequently resolved the matching problem through the introduction of a new set of modes for which h provides the lateral, as well as the vertical, scale; however, the resulting perturbation expansion remains singular (in the sense that the scale of the solution in the neighbourhoods of the inertial latitudes differs intrinsically from that of the basic solution).

Stewartson & Rickard associated the fact that the limit $\delta \downarrow 0$ is singular for $\lambda < 1$ with the fact that the partial differential equation that governs small harmonic disturbances in a rotating, non-diffusive, homogeneous fluid is hyperbolic for $\sigma < 2\omega$, whereas the boundary conditions are prescribed over closed surfaces, which implies that the problem is not well posed. This difficulty, which appears to have been recognized originally by Hadamard (1936) and Solberg (1936*b*) (see also Weinstein 1942; Bateman 1943; Görtler 1943, 1957), is compounded by stratification, for which the corresponding partial differential equation is elliptic/hyperbolic for

$$\sigma^2 - (4\omega^2 + N^2) + (N/\sigma)^2 f^2 \gtrless 0, \tag{1.9}$$

where f is defined by (1.8). It follows that the boundary-value problem for tidal oscillations is well posed if and only if

$$\sigma > (4\omega^2 + N^2)^{\frac{1}{2}} \equiv \sigma_*, \tag{1.10}$$

which, for typical N, excludes all frequencies that are *tidal* in the conventional sense of the term. The equation is hyperbolic for all latitudes if

$$\text{either} \qquad 2\omega < \sigma < N_{\min} \quad \text{or} \quad N_{\max} < \sigma < 2\omega, \tag{1.11a,b}$$

where $N_{\min/\max}$ is the minimum/maximum value of N. The equation is mixed if

$$\text{either} \qquad 0 < \sigma < \min(2\omega, N) \quad \text{or} \quad \max(2\omega, N) < \sigma < \sigma_*. \tag{1.12a,b}$$

The regime (1.12*b*), which represents a rotation-induced broadening of the internal-wave cut-off (which is given by $\sigma = N$ if $\omega = 0$), is of limited interest in the present context. The regimes described by (1.9)–(1.12) are depicted in figure 1.

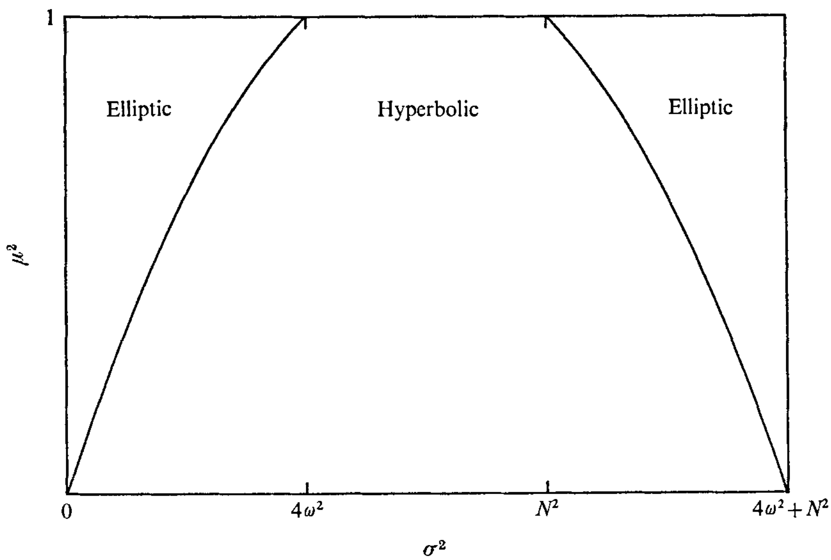

FIGURE 1. The regimes delineated by (1.9)–(1.12). The drawing is scaled for $N^2 = 8\omega^2$ in the interest of clarity, whereas $N^2 = 800\omega^2$ would be more realistic for the oceans. The tidal regime is $\sigma^2 < 4\omega^2$, in which the governing partial differential equation is mixed.

The preceding discussion implies that the critical limits for LTE are $\Gamma \downarrow 0$ and $\delta \downarrow 0$. The perturbation problem associated with these limits is attacked below according to the following plan. The equations of motion and the boundary conditions for a non-diffusive global ocean of uniform depth are derived in § 2 on the basis of (ii)–(v) above and, in addition, the neglect of the radial variation of the metrical coefficients. A more complete analysis reveals that the effects of this last approximation are uniformly $O(\delta)$ and therefore no more significant than those of ellipticity and the radial variation of gravity in the present context. [Phillips (1966, 1968) argues from angular momentum considerations that the radial variation of the metrical coefficients and the horizontal component of the Earth's rotation are of strictly comparable significance for a homogeneous ocean but appears to agree with Veronis (1968) that the argument may not be physically conclusive. He also recognizes that it is not applicable to a stratified fluid if $\mathcal{N}^2 \gg 1$.] The eigenvalue problem implied by § 2 for finite δ, Γ and s is examined in § 3 with β as the eigenparameter (for given λ). It is demonstrated that the eigenvalues must be real, but not necessarily positive, and that the eigenfunctions are orthogonal. LTE are derived in § 4 by letting Γ, $\delta \downarrow 0$ with β and λ fixed and $s = O(1)$, and the properties of the resulting normal modes are recapitulated.

A joint perturbation expansion in Γ and δ with $s = O(1)$ is found to be singular at $\mu = \pm\lambda$ and does not appear to admit a matched inner expansion that is physically acceptable.† This fundamental difficulty appears to follow from the

† I have carried out a detailed analysis for the special case of constant N. The outer approximation for the pressure and velocity is similar to that in § 6 below, but the corresponding result for the vertical displacement is significantly different. The inner approximation leads to Tricomi's (1923) equation.

failure of LTE to provide an adequate approximation to the characteristics of the primitive equations in the hyperbolic domain if $\lambda < 1$. A consideration of these characteristics suggests that a uniformly valid approximation for $\mathcal{N} \gg 1$ can be obtained by retaining buoyancy in the basic approximation. This leads (in § 5) to a formulation that closely resembles that for atmospheric tides (Chapman & Lindzen 1970, chap. 3) and permits solution by separation of variables. The vertical-structure modes, which form a complete set, are characterized by the eigenvalues $\alpha_0 = O(1)$ and $\alpha_j = O(1/s)$ for $j = 1, 2, \ldots, \infty$. The corresponding level-surface modes satisfy LTE with β replaced by $\alpha_j \beta$ and may be expanded in the normal modes of LTE. That component of the solution corresponding to α_0 represents the depth-averaged, or barotropic, motion, which has lateral and radial scales of a and h, respectively, and is correctly described by LTE. That component corresponding to $\alpha_j, j \geqslant 1$, represents internal-wave, or baroclinic, motion which has lateral and radial scales of $s^{\frac{1}{2}}a$ and h, respectively, and has been studied previously by Hughes (1964) and Munk & Phillips (1968) and, in the context of equatorial trapping, by Blandford (1966), Matsuno (1966) and Munk & Moore (1968). The barotropic and baroclinic motions are uncoupled in the limit $\delta \downarrow 0$, but a perturbation expansion (in § 6) reveals that the Coriolis acceleration associated with the horizontal component of the Earth's rotation induces a qualitatively significant coupling if and only if $\lambda < 1$. The vertical acceleration also introduces such a coupling, but it is negligible compared with the Coriolis coupling.

The inequality $\alpha_j \beta \gg 1$ for the baroclinic motion invites an asymptotic approximation, which is developed in § 7. The resulting solutions have Airy turning points at $\mu = \pm \lambda$ and are trapped in $|\mu| < \lambda$. The spectrum is discrete but dense, and it appears that resonant amplification of the baroclinic modes that are forced (through Coriolis coupling) by the barotropic modes could imply internal displacements that dominate those of the basic motion.

The reader who is interested primarily in the essential results, and not in the details of the limit process, could omit §§ 5–7 and pass directly from § 4 to § 8, where the equations necessary for a uniformly valid formulation of the tidal equations are recapitulated.

2. Equations of motion

We consider free oscillations in a global ocean of uniform depth. The assumption of a perfect fluid implies the equations of motion

$$D\mathbf{q}/Dt + 2\boldsymbol{\omega} \times \mathbf{q} = -\rho^{-1}\nabla p - \nabla \psi, \quad (2.1a)$$

$$\nabla . \mathbf{q} = -\rho^{-1} D\rho/Dt \quad (2.1b)$$

and

$$Dp/Dt = c^2 D\rho/Dt, \quad (2.1c)$$

where $\mathbf{q}$ is the particle velocity in the rotating reference frame; $\boldsymbol{\omega}$ is the angular velocity of the Earth; p, ρ and c are pressure, density and sonic velocity; ψ is the geopotential. The assumptions of a spherical Earth and a uniform gravitational field imply

$$\psi = g(r-a). \quad (2.2)$$

The equilibrium state (subscript zero) may be described by the density $\rho_0(r)$ and the hydrostatic equation

$$p_0'(r) = -g\rho_0(r). \tag{2.3}$$

We consider small oscillations about this equilibrium state of the form

$$\{p-p_0, \rho-\rho_0, \zeta\} = \hbar\mathscr{R}[\{\rho_0 ghP, \rho_0 R, hZ\}\, e^{i(\sigma t+m\phi)}] \tag{2.4a}$$

and
$$\mathbf{q} = \{u, v, w\} = \hbar(gh/2\omega a)\,\mathscr{R}[\{-iU, V, iW\}\, e^{i(\sigma t+m\phi)}], \tag{2.4b}$$

where $\hbar h$ is the vertical scale of the oscillation; u, v and w are the velocity components in the directions of increasing θ (south), ϕ (east) and r (vertical), respectively; σ is the angular frequency and m is the azimuthal wavenumber; $\mathscr{R}$ implies *the real part of*; P, R, U, V, W and Z are dimensionless functions of μ and either $\mathfrak{r}$ or z (we ultimately use z)†;

$$\mu = \cos\theta, \quad \mu_* = \sin\theta, \quad z = (r-a)/h \equiv (\mathfrak{r}-1)/\delta. \tag{2.5a, b, c}$$

Substituting (2.2)–(2.5) into (2.1), neglecting the radial variation of the metrical coefficients and linearizing in $\hbar$, we obtain (cf. Eckart 1960, § 38)

$$\begin{bmatrix} \lambda & -\mu & 0 \\ -\mu & \lambda & \mu_* \\ 0 & \mu_* & \lambda-\lambda^{-1}\mathscr{N}^2 \end{bmatrix} \begin{bmatrix} U \\ V \\ W \end{bmatrix} = \begin{bmatrix} \mu_*\partial_\mu \\ -m/\mu_* \\ \partial_{\mathfrak{r}} - S \end{bmatrix} P, \tag{2.6a}$$

$$(\mu_* U)_\mu + (m/\mu_*)\, V + W_{\mathfrak{r}} = \beta\Gamma\lambda(Z-P), \tag{2.6b}$$

$$R = \Gamma P + sZ, \quad W = \beta\delta\lambda Z, \tag{2.6c, d}$$

where
$$\Gamma = gh/c_0^2, \quad S \equiv s/\delta = -(\log\rho_0)_{\mathfrak{r}} - (ga/c_0^2) \equiv N^2 a/g \equiv \beta\delta\mathscr{N}^2, \tag{2.7a, b}$$

$$\beta = 4\omega^2 a^2/gh, \quad \delta = h/a, \quad \lambda = \sigma/2\omega, \quad \mathscr{N} = N/2\omega, \tag{2.8a–d}$$

c_0 is the equilibrium value of c (corresponding to p_0 and ρ_0), and the subscripts μ and $\mathfrak{r}$ imply partial differentiation. The boundary conditions at the free surface ($p = p_0$ at $r = a+\zeta$) and rigid bottom ($\zeta = 0$ at $r = a-h$) imply

$$P-Z = 0 \quad (\mathfrak{r} = 1 \text{ or } z = 0), \quad Z = 0 \quad (\mathfrak{r} = 1-\delta \quad \text{or} \quad z = -1). \tag{2.9a, b}$$

We also require P and Z to be finite at $\mu = \pm 1$.

Eliminating U, V, W and Z from (2.6) yields a second-order, self-adjoint partial differential equation for P that is elliptic/hyperbolic for $D \lessgtr 0$, where [cf. (1.9)]

$$D = \lambda^2\lambda_*^2 + \mathscr{N}^2(\lambda^2-\mu^2), \quad \lambda_*^2 = 1-\lambda^2. \tag{2.10a, b}$$

3. Eigenvalue problem

We now consider the eigenvalue problem posed by (2.6) and (2.9) for given λ and m with β as the eigenparameter. It follows from (2.6) and (2.9) that each of P, U, V, W and Z must be either even or odd in μ and that V, W and Z have the same symmetry as P, whilst U has the opposite symmetry. It suffices, in the

† Choosing $\log\mathfrak{r}/\log(1-\delta)$, rather than $(\mathfrak{r}-1)/\delta$, as the stretched vertical co-ordinate would offer some advantages, but they are unimportant in the present context.

present context, to consider only positive λ; the solution for negative λ and given m may be obtained from the solution for $|\lambda|$ and $-m$ by changing the signs of U and W (but not Z).

Let $\{P, U, V, W, Z; \beta\}$ be a solution of (2.6) and (2.9) and let

$$\{P^*, U^*, V^*, W^*, Z^*; \beta^*\}$$

be a corresponding solution of, say, (2.6)* and (2.9)* for the same values of m and λ. Multiplying (2.6a)* and (2.6b) through by $\rho_0\{U, V, W\}$ and $\rho_0 P^*$, respectively, adding the results, integrating over $-1 < \mu < 1$ and $1-\delta < r < 1$, and invoking (2.9a, b) to simplify the r-integral of $\partial_r(\rho_0 P^* W)$ yields (after some reduction and the invocation of (2.6d) and (2.7))

$$\begin{aligned}\delta\beta\lambda\left\{\int_{-1}^{1}(\rho_0 ZZ^*)_a\,d\mu+\int_{1-\delta}^{1}\rho_0 S\,dr\int_{-1}^{1}ZZ^*\,d\mu\right\}\\ =\int_{1-\delta}^{1}\rho_0\,dr\int_{-1}^{1}\{\lambda(UU^*+VV^*+WW^*-\Gamma PP^*)\\ -\mu(UV^*+U^*V)+\mu_*(VW^*+V^*W)\}\,d\mu.\end{aligned} \tag{3.1}$$

Interchanging Z and Z^*, etc. in (3.1) and taking the difference of the two results on the hypothesis that $\beta \neq \beta^*$ yields

$$\int_{-1}^{1}(\rho_0 ZZ^*)_a\,d\mu+\int_{1-\delta}^{1}\rho_0 S\,dr\int_{-1}^{1}ZZ^*\,d\mu = 0 \quad (\beta \neq \beta^*), \tag{3.2}$$

from which it follows that Z and Z^* are orthogonal in the indicated sense. We remark that regarding λ, rather than β, as the eigenvalue yields a more complicated orthogonality.

Now suppose that $\{P^*, \ldots; \beta^*\}$ is the complex conjugate of $\{P, \ldots; \beta\}$. Each of the integrals in (3.1) then is real, from which it follows that β also must be real for any non-trivial eigensolution; however, β is not necessarily positive. (It also follows from the the preceding argument that λ, *qua* eigenvalue for given β, must be real.) The eigensolution itself may be complex, but only within a complex constant that is common to each of P, U, V, W and Z.

4. Laplace's approximation

Substituting $r = 1+\delta z$ into the third row of (2.6a) and letting $\delta \downarrow 0$ with β and $\mathcal{N}$ fixed and $s = O(\delta)$ yields

$$P_z = 0. \tag{4.1}$$

This, together with (2.9a, b), suggests a solution of the form

$$P = H(\mu), \quad Z = (1+z)H(\mu). \tag{4.2a, b}$$

Substituting W from (2.6d) and P and Z from (4.2a, b) into the first two rows of (2.6a) and (2.6b), letting $\Gamma, \delta \downarrow 0$ with β fixed, and combining the results, we obtain

$$\begin{bmatrix}\lambda & -\mu & -\mu_*\partial_\mu\\ -\mu & \lambda & m/\mu_*\\ \partial_\mu\mu_* & m/\mu_* & \beta\lambda\end{bmatrix}\begin{bmatrix}U\\ V\\ H\end{bmatrix} \equiv \mathbf{L}\{U, V, H; \beta\} = 0. \tag{4.3}$$

Eliminating U and V from (4.3) yields

$$\begin{bmatrix} U \\ V \end{bmatrix} = \frac{1}{\lambda^2-\mu^2}\begin{bmatrix} \lambda & \mu \\ \mu & \lambda \end{bmatrix}\begin{bmatrix} \mu_* \partial_\mu \\ -m/\mu_* \end{bmatrix} P \equiv \mathscr{U} H \tag{4.4}$$

and

$$(\mathscr{L}+\beta)H = 0, \tag{4.5}$$

where

$$\mathscr{L} = \partial_\mu \not{p} \partial_\mu - \frac{m}{\lambda}\left(\frac{\mu}{\lambda^2-\mu^2}\right)_\mu - \frac{m^2}{\mu_*^2(\lambda^2-\mu^2)}, \quad \not{p} = \frac{\mu_*^2}{\lambda^2-\mu^2}. \tag{4.6a, b}$$

We note that (4.4) is indeterminate at $\mu = \pm\lambda$, where

$$(\mu_*^2\,\partial_\mu \mp m)H = 0 \quad (\mu = \pm\lambda). \tag{4.7}$$

The differential equation (4.5), often designated as Laplace's tidal equation, was integrated originally by Laplace and then by Hough (Lamb 1932, §§ 221, 222); it has recently been studied by Flattery (1967) and Longuet-Higgins (1968). It has apparent singularities at $\mu = \pm\lambda$, regular singularities at $\mu = \pm 1$, and an irregular singularity at $\mu = \infty$. It admits a complete set of eigensolutions (usually designated as Hough functions) for given m and λ with β as the eigenvalue, and is exceptional, *vis-à-vis* the typical Sturm–Liouville equation, in the existence of negative eigenvalues if and only if $\lambda < 1$. The eigensolutions are regular in $|\mu| < 1$, $O(\mu_*^m)$ as $\mu \to \pm 1$, mutually orthogonal in $(-1, 1)$ and may be assumed to be real. We denote the complete set of eigensolutions by $\{H_n, U_n, V_n; \beta_n, \lambda, m\}$ or, more briefly, $\{H_n; \beta_n\}$, and choose the normalization

$$\int_{-1}^{1} H_n^2\, d\mu = 1. \tag{4.8}$$

Sturm's comparison theorems do not hold for the H_n if $\lambda < 1$. The case $\lambda = 1$ is special (Eckart 1960, § 99).

5. Separation of variables

Laplace's approximation provides a complete set of expansion functions, $\{H_n; \beta_n\}$, for the level surfaces but not for the vertical structure. It also implies the trivial approximation $d\mu/dz = 0$ for the characteristics of (2.6) in the hyperbolic domain. Inspection reveals that if $\mathscr{N}^2 \gg \lambda^2$ the only modification of (4.1) that is required to obtain both a complete set of vertical-structure functions and a significant approximation to the characteristics is to retain the buoyancy term $-(\mathscr{N}^2/\lambda)W$ in the third row of (2.6*a*); however, we also retain the stratification term SP in (2.6*a*) and the compressibility term $\Gamma(Z-P)$ in (2.6*b*) in order to demonstrate that their effects are uniformly small in the limit $\Gamma, \delta \downarrow 0$.† Invoking (2.5*c*), (2.6*d*) and (2.7*b*) for Z, W and s then yields

$$\begin{bmatrix} \lambda & -\mu \\ -\mu & \lambda \end{bmatrix}\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} \mu_* \partial_\mu \\ -m/\mu_* \end{bmatrix} P, \tag{5.1a}$$

$$sZ = sP - P_z \tag{5.1b}$$

and

$$(\mu_* U)_\mu + (m/\mu_*)\, V + \beta\lambda\{Z_z - \Gamma(Z-P)\} = 0, \tag{5.1c}$$

† Neglecting SP and $\Gamma(Z\text{–}P)$ is equivalent to the Boussinesq approximation. Equations (5.1), which retain these terms, resemble the corresponding equations for atmospheric tides (Chapman & Lindzen 1970, chap. 3); however, the boundary conditions (5.2) differ significantly from those for atmospheric tides.

which differ from (2.6a, b) only in the omission of the Coriolis-acceleration terms $\mu_* V$ and $\mu_* W$ and the vertical-acceleration term λW (these terms will be included in § 6). Substituting (5.1b) into (2.9) yields the boundary conditions

$$P_z = 0 \quad (z = 0), \qquad P_z - sP = 0 \quad (z = -1). \tag{5.2a, b}$$

Eliminating U, V and Z from (5.1) yields

$$\mathscr{L}P = (\beta/\rho_0)\{(\rho_0/s)P_z\}_z, \tag{5.3}$$

where $\mathscr{L}$ is defined by (4.6).

The eigenvalue problem posed by (5.1a), (5.2) and (5.3) admits separable solutions of the form

$$\{P, U, V\} = F_j(z)\{H_n(\mu), U_n(\mu), V_n(\mu)\}, \quad \beta = \beta_n/\alpha_j, \tag{5.4a, b}$$

where $\{H_n(\mu); \beta_n\}$ are the eigenfunctions of § 4, and $\{F_j(z); \alpha_j\}$ are the orthonormal eigenfunctions determined by the Sturm–Liouville problem

$$(\hat{\rho}F'/s)' + \alpha\hat{\rho}F = 0, \tag{5.5}$$

$$F' = 0 \quad (z = 0), \qquad F' - sF = 0 \quad (z = -1), \tag{5.6a, b}$$

$$\langle \hat{\rho}F^2 \rangle = 1, \quad \hat{\rho}(z) = \rho_0/\langle \rho_0 \rangle, \tag{5.7a, b}$$

in which the primes imply differentiation with respect to z and $\langle\ \rangle$ implies integration over $(-1, 0)$. Letting $\Gamma \downarrow 0$ with $s = O(\Gamma)$, we obtain

$$\alpha_0 = \hat{\rho}(-1) - \langle z^2 s \rangle + O(\Gamma^2), \quad F_0 = 1 - \langle z^2 s \rangle - \int_{-1}^{z} zs\,dz + O(\Gamma^2), \tag{5.8a, b}$$

$$\alpha_j = O(1/s), \quad \langle F_j \rangle = O(s) \quad (j \geqslant 1), \tag{5.9a, b}$$

where, by definition, $0 < \alpha_0 < \alpha_1 \ldots$.

The solution given by (5.4) for $j = 0$ is essentially the depth-averaged motion and represents a surface wave; it may be described as barotropic and is equivalent to the solution of § 4 within $O(\Gamma)$. The higher modes ($j \geqslant 1$) represent internal waves and may be described as baroclinic. They are determined within factors of $1 + O(\Gamma)$ by

$$(s^{-1}F')' + \alpha F = 0, \tag{5.10a}$$

$$F' = 0 \quad (z = 0, -1), \quad \langle F^2 \rangle = 1 \tag{5.10b, c}$$

in place of (5.5)–(5.7). Moreover, F_j may be neglected compared with F_j'/s to this same approximation, and (5.1b) and (5.4a) then imply

$$Z = \alpha_j G_j(z) H_n(\mu), \quad G_j = -(\alpha_j s)^{-1} F_j' \quad (j \geqslant 1), \tag{5.11a, b}$$

where the G_j form a complete orthonormal set and satisfy

$$G'' + \alpha s G = 0, \tag{5.12a}$$

$$G = 0 \quad (z = 0, -1), \quad \langle sG^2 \rangle = 1/\alpha. \tag{5.12b, c}$$

Characteristics

The characteristics of (2.6) in the hyperbolic domain are determined by

$$d\mu/d\tau = \mu_*(\lambda^2 - \mu^2)^{-1}[-\mu\mu_* \pm \{\lambda^2\lambda_*^2 + \mathscr{N}^2(\lambda^2 - \mu^2)\}^{\frac{1}{2}}]. \tag{5.13}$$

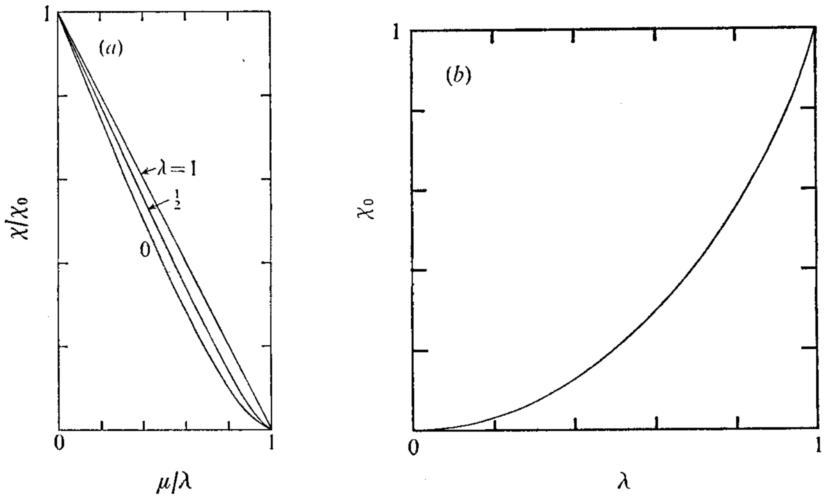

FIGURE 2. (a) $\chi(\mu,\lambda)$, as given by (5.16a), and (b) $\chi_0(\lambda)$, as given by (5.16b).

The characteristics of either (5.1) or (5.3), for which the hyperbolic domain is $\mu^2 < \lambda^2$, are determined by

$$d\mu/dz = \pm\delta\mathcal{N}\not{p}^{\frac{1}{2}} \equiv \pm(\not{p}s/\beta)^{\frac{1}{2}}, \tag{5.14}$$

where $\not{p}$ is given by (4.6b). Integrating (5.14) yields the characteristic co-ordinates

$$\gamma_{\pm} = \chi(\mu,\lambda) \mp \delta\int_0^z \mathcal{N}(z)\,dz, \tag{5.15}$$

where

$$\chi(\mu,\lambda) = \chi_0(\lambda) - E(\sin^{-1}(\mu/\lambda),\lambda) + \lambda_*^2 F(\sin^{-1}(\mu/\lambda),\lambda), \tag{5.16a}$$

$$\chi_0(\lambda) \equiv \chi(0,\lambda) = E(\tfrac{1}{2}\pi,\lambda) - \lambda_*^2 F(\tfrac{1}{2}\pi,\lambda), \tag{5.16b}$$

and E and F are elliptic integrals of the second and first kind; χ and χ_0 are plotted in figures 2(a) and (b).

WKB approximations

It follows from (5.4b) and (5.9a) that $\beta_n \gg 1$ if $j \geqslant 1$ and $\beta = O(1)$. The asymptotic solution of (4.3) as $\beta_n \to \infty$ is considered in §7 and yields the WKB approximations [uniformly valid approximations are given by (7.16)]

$$H_n \sim \chi_0^{-\frac{1}{2}}\not{p}^{-\frac{1}{4}}\cos(\beta_n^{\frac{1}{2}}\chi + \tfrac{1}{4}\pi) \quad (\beta_n\chi \to \infty), \tag{5.17a}$$

$$(\mu_*/\lambda)\,U_n \sim (\mu_*/\mu)\,V_n \sim (\beta_n/\chi_0)^{\frac{1}{2}}\not{p}^{\frac{1}{4}}\sin(\beta_n^{\frac{1}{2}}\chi + \tfrac{1}{4}\pi) \tag{5.17b}$$

and

$$\beta_n \sim \{(n-\tfrac{1}{2})\pi/2\chi_0\}^2 \quad (n \gg 1), \tag{5.17c}$$

where n is the number of zeros of H_n in $(-\lambda,\lambda)$, and χ and χ_0 are given by (5.16); the solutions have no zeros, and decay exponentially, in $|\mu| > \lambda$. The corresponding (Liouville) solution of (5.10),

$$F_j \sim \left(\frac{2\mathcal{N}}{\langle\mathcal{N}\rangle}\right)\cos\left\{\frac{j\pi}{\langle\mathcal{N}\rangle}\int_0^z \mathcal{N}\,dz\right\}, \quad \alpha_j\beta = \left(\frac{j\pi}{\delta\langle\mathcal{N}\rangle}\right)^2, \tag{5.18a, b}$$

is quantitatively valid only for $j \gg 1$ but is likely to be at least qualitatively valid for all $j \geqslant 1$. Substituting (5.17c) and (5.18b) into (5.4b) yields

$$\delta\langle\mathcal{N}\rangle/\chi_0(\lambda) = 2j/(n-\tfrac{1}{2}) \quad (n \gg j \gg 1) \tag{5.19}$$

for internal-wave resonance; the approximation is likely to provide useful estimates for all j ($n \gg j$ is automatically satisfied for $\delta\langle\mathcal{N}\rangle \ll 1$). Combining (5.17$a, b$) and (5.18$a$) in (5.4$a$) yields results that may be expressed in terms of the characteristic co-ordinates (5.15).

6. Perturbation solution

We now regard the acceleration terms $\mu_* W$ and $\mu_* V + \lambda W$† in (2.6) as forcing functions for a perturbation solution of the form

$$\{P, U, V\} = \sum_j F_j(z)\{\mathcal{P}_j(\mu), \mathcal{U}_j(\mu), \mathcal{V}_j(\mu)\}, \tag{6.1a}$$

where the summation is over the complete set of the orthogonal eigenfunctions determined by (5.5)–(5.7), and

$$\{\mathcal{P}, \mathcal{U}, \mathcal{V}\} = \langle\hat{\rho}F\{P, U, V\}\rangle \tag{6.1b}$$

is the corresponding finite transform of $\{P, U, V\}$. The amplitude of the vertical displacement is determined by the third row of (2.6a), which may be rewritten in the form [cf. (5.1b)]

$$Z = P - s^{-1}P_z + (\delta/s)(\mu_* V + \lambda W). \tag{6.2}$$

Transforming the first two rows of (2.6a), (6.2) and (2.6b) by multiplying through by $\hat{\rho}F$, $\hat{\rho}(F' - sF)$ and $\hat{\rho}F$, respectively, integrating over $(-1, 0)$ and simplifying the results through integration by parts, (2.7), (2.9) and (5.5)–(5.7), we obtain (see appendix for details)

$$\mathbf{L}\{\mathcal{U}, \mathcal{V}, \mathcal{P}; \alpha\beta\} = \beta\delta\lambda\mathbf{Q}, \tag{6.3}$$

where
$$\mathbf{Q} = \{0, -\mu_*\langle\hat{\rho}FZ\rangle, \langle\hat{\rho}(s^{-1}F' - F)(\mu_* V + \lambda W)\rangle\} \tag{6.4}$$

and $\mathbf{L}$ is defined by (4.3).

Equation (6.3), which implies solutions of the form (5.4) for $\beta\delta \downarrow 0$, may be solved by iteration. We consider here the first iteration about the nth barotropic mode, for which $Z = (1+z)H_n$, $V = V_n$, and λW_n may be neglected compared with $\mu_* V_n$. Approximating $\hat{\rho}$ by 1 and appending the subscripts j and n in (6.4), we obtain

$$\mathbf{Q}_{jn} = \mu_*\{0, -\mathcal{A}_j H_n, \mathcal{B}_j V_n\}[1 + O(\Gamma, \beta\delta)], \tag{6.5}$$

where
$$\mathcal{A}_j = \langle(1+z)F_j\rangle, \quad \mathcal{B}_j = \langle s^{-1}F_j' - F_j\rangle. \tag{6.6a, b}$$

† The vertical-acceleration term λW could have been retained in (5.1b), thereby leaving only the Coriolis-acceleration terms $\mu_* V$ and $\mu_* W$ as perturbation forcing functions; however, this offers no advantage if $\mathcal{N}^2 \gg \lambda^2$ and would have led to more complicated $F_j(z)$.

Invoking (5.8*b*) for $j = 0$ and multiplying (5.5) through by $1+z$, integrating over $(-1, 0)$, and invoking (5.9*b*) and (5.11) for $j > 0$ yields

$$\mathscr{A}_0 = -\mathscr{B}_0 = \tfrac{1}{2}, \quad \mathscr{A}_j = \alpha_j^{-1}\mathscr{B}_j = -\langle G_j \rangle \quad (j > 0) \tag{6.7a,b}$$

within $O(\Gamma)$. Substituting (6.5) and (6.7) into (6.3) and approximating β by β_n for $j > 0$ (but not for $j = 0$; see below) yields

$$\mathbf{L}\{\mathscr{U}_0, \mathscr{V}_0, \mathscr{P}_0; \alpha_0\beta\} = -\tfrac{1}{2}\beta_n \delta\lambda\mu_*\{0, H_n, V_n\} \tag{6.8a}$$

and
$$\mathbf{L}\{\mathscr{U}_j, \mathscr{V}_j, \mathscr{P}_j; \alpha_j\beta_n\} = \mathscr{A}_j\beta_n\delta\lambda\mu_*\{0, -H_n, \alpha_j V_n\} \quad (j > 0). \tag{6.8b}$$

Different techniques are expedient for the solution of (6.8*a*) and (6.8*b*), both because $\alpha_j \gg 1$ for $j > 0$ and because $\{U_n, V_n, H_n\}$ is an eigensolution of the operator $\mathbf{L}$ in the limit $\alpha_0\beta \to \beta_n$. [The solution of either (6.8*a*) or (6.8*b*) could be obtained by eliminating $\mathscr{U}$ and $\mathscr{V}$ and expanding $\mathscr{P}$ in the complete set $\{H_l; \beta_l\}$, but the expansions are not uniformly convergent near $\mu = \pm\lambda$ and are inefficient for $j > 0$.] Considering first (6.8*a*), we eliminate $\mathscr{U}_0$ and $\mathscr{V}_0$ and pose

$$\mathscr{P}_0 = H_n + \beta_n\delta\mathscr{H}_n, \quad \alpha_0\beta = \beta_n(1+\delta b_n) \tag{6.9a,b}$$

to obtain
$$\{\mathscr{U}_0, \mathscr{V}_0\} = \mathscr{U}\mathscr{P}_0 - \tfrac{1}{2}\beta_n\delta\lambda\mu_*(\lambda^2-\mu^2)^{-1}\{\mu, \lambda\} H_n \tag{6.10}$$

and
$$(\mathscr{L}+\beta_n)\mathscr{H}_n = \tfrac{1}{2}(\partial_\mu \mu\mu_*^2 + m\lambda)(\lambda^2-\mu^2)^{-1}H_n - \tfrac{1}{2}\mu_* V_n - b_n H_n, \tag{6.11}$$

where $\mathscr{U}$ and $\mathscr{L}$ are defined by (4.4) and (4.6).

The solution of (6.11), subject to finiteness conditions at $\mu = \pm 1$, exists if and only if the right-hand side is orthogonal to H_n. Invoking this condition, integrating the term $H_n\partial_\mu\ldots$ by parts, and invoking (4.4) for V_n and the normalization (4.8) for H_n, we obtain

$$b_n = -\int_{-1}^{1} \mu_* H_n V_n \, d\mu. \tag{6.12}$$

Solving (6.11) by a modification of the method of variation of parameters, in which the path of integration is taken from $\mu = 1$ and temporarily deformed into the complex plane to avoid the apparent singularities at $\mu = \pm\lambda$ (if $\lambda^2 < 1$) and any zeros of H_n and the first term on the right-hand side is integrated by parts, we obtain

$$\mathscr{H}_n = -H_n\left\{\tfrac{1}{4}\mu_*^2 + \int_\mu^1 (\not\!\mu H_n^2)^{-1} d\mu \int_\mu^1 (\mu_* H_n V_n + b_n H_n^2)\, d\mu\right\}, \tag{6.13}$$

which is regular at $\mu = \pm\lambda$ and the zeros of H_n and $O(\mu_*^2 H_n)$ as $\mu \to \pm 1$. The corresponding approximations to $\mathscr{U}_0$ and $\mathscr{V}_0$, obtained from (6.9*a*) and (6.10), also are regular at $\mu = \pm\lambda$ and $O(U_n)$ as $\mu \to \pm 1$.

The asymptotic solution of (6.8*b*) for $\lambda < 1$ is complicated by the turning points of the operator $\mathbf{L}$ (and also of $\mathscr{L}$) at $\mu = \pm\lambda$. A uniformly valid approximation for $\lambda < 1$ is developed in the following section; however, it is expedient to consider first the solution for either $\lambda^2 < \mu^2 < 1$ or $\lambda > 1$, which may be expanded in inverse powers of $\alpha_j\beta_n$. The dominant terms in this expansion are given by

$$\{\mathscr{P}_j, \mathscr{U}_j, \mathscr{V}_j\} \sim \mathscr{A}_j\delta\{\mu_* V_n, \mathsf{U}_n, \mathsf{V}_n\} \tag{6.14}$$

and

$$\{\mathsf{U}_n, \mathsf{V}_n\} = \mathscr{U}\mu_* V_n - \beta_n \lambda\mu_*(\lambda^2-\mu^2)^{-1}\{\mu,\lambda\} H_n \tag{6.15a}$$

$$= (\mu^2-\lambda^2)^{-1}[\beta_n\mu_*\{2\lambda\mu, \lambda^2+\mu^2\} H_n + (m-\lambda^{-1}\mu_*^2)\{\lambda,\mu\} U_n + m\{\mu,\lambda\} V_n], \tag{6.15b}$$

where $\mathscr{U}$ is defined by (4.4), and (6.15b) follows from (6.15a) with the aid of the identities implied by (4.3). Substituting (6.14) into (6.1a), remarking that

$$\sum_{j=1}^{\infty} \mathscr{A}_j F_j(z) \equiv \sum_{n=0}^{\infty} \langle (1+z) F_j\rangle F_j(z) - \langle (1+z) F_0\rangle F_0(z) \tag{6.16a}$$

$$= 1+z-\langle 1+z\rangle\{1+O(\Gamma)\} \tag{6.16b}$$

$$= \tfrac{1}{2}+z+O(\Gamma) \tag{6.16c}$$

and neglecting terms of second order in Γ and δ, we obtain

$$\{P, U, V\} \sim F_0(z)\{\mathscr{P}_0(\mu), \mathscr{U}_0(\mu), \mathscr{V}_0(\mu)\} + \delta(\tfrac{1}{2}+z)\{\mu_* V_n(\mu), \mathsf{U}_n(\mu), \mathsf{V}_n(\mu)\}. \tag{6.17a}$$

The corresponding approximation to Z, obtained by substituting (6.9) and (6.17) into (6.2), is

$$Z = \{F_0(z) - s^{-1}F_0'(z)\} H_n(\mu) + \beta_n \delta(1+z)\mathscr{H}_n(\mu). \tag{6.17b}$$

We emphasize that (6.17a,b) are valid only if either $\mu^2-\lambda^2 \gg \delta^2\mathscr{N}^2$ or $\lambda^2-1 \gg \delta^2 N^2$ and that they then differ from Laplace's approximation (§ 4) by $O(\Gamma, \beta_n\delta)$; moreover, the $O(\delta)$ terms in (6.17a) then may not be significant in consequence of the *a priori* neglect of other $O(\delta)$ effects.

7. Asymptotic solution for internal waves

We now seek a uniformly valid solution of (6.8b) in $0 \leqslant \mu < 1$ as $\alpha\beta \to \infty$ (we temporarily drop the subscripts j and n) with

$$\lambda, 1-\lambda \gg 1/\kappa, \quad m \ll \kappa, \quad \kappa \equiv (\alpha\beta)^{\frac{1}{3}}. \tag{7.1a,b,c}$$

The corresponding solution in $-1 < \mu \leqslant 0$ follows from symmetry.

A preliminary investigation reveals that the asymptotic determination of $\mathscr{U}$ is somewhat simpler than that of $\mathscr{P}$. Eliminating $\mathscr{P}$ and $\mathscr{V}$ from (6.8b), letting $\alpha\beta \to \infty$ and invoking (7.1), we obtain

$$\mu_*(\mu_*\mathscr{U})'' + \alpha\beta(\lambda^2-\mu^2)\mathscr{U} = \mathscr{A}\alpha\beta\delta(\lambda^2-\mu^2)\,\mathsf{U}(\mu), \tag{7.2}$$

where U is given by (6.15) and the primes imply differentiation with respect to μ. The boundary conditions (which replace the finiteness conditions at $\mu = \pm 1$) are

$$\mathscr{U} \sim \mathscr{A}\delta\mathsf{U}, \quad \alpha\beta(\mu^2-\lambda^2) \to \infty \tag{7.3a}$$

and either

$$\mathscr{U} = 0 \quad \text{or} \quad \mathscr{U}' = 0 \quad (\mu = 0), \tag{7.3b,c}$$

according as $\mathscr{U}(\mu)$ is odd or even, respectively. The approximation is not uniform as $\mu \to \pm 1$, but this is unimportant if (7.1a,b) are satisfied.

A uniformly valid solution of (7.2) and (7.3) with a relative error $O(1/\kappa)$ [the relative error in (7.2), *vis-à-vis* (6.8*b*), is $O(1/\alpha)$] may be obtained by Langer's method. Invoking the transformation

$$\mu_* \mathscr{U}(\mu) = \mathscr{A}\delta\lambda(-\not{p}\phi)^{\frac{1}{4}}\,\Phi(\phi), \quad \mu_*^{-1}(\mu^2-\lambda^2)\,\mathsf{U}(\mu) = \lambda(-\not{p}\phi)^{-\frac{3}{4}}\Psi(\phi), \quad (7.4a,b)$$

$$\tfrac{2}{3}(-\phi)^{\frac{3}{2}} = \int_\mu^\lambda \{\not{p}(t)\}^{-\frac{1}{2}}dt = \chi(\mu,\lambda) \quad (0 \leqslant \mu \leqslant \lambda), \tag{7.5a}$$

$$\tfrac{2}{3}\phi^{\frac{3}{2}} = \int_\lambda^\mu \{-\not{p}(t)\}^{-\frac{1}{2}}dt = \chi(\mu_*,\lambda_*) \quad (\lambda \leqslant \mu \leqslant 1), \tag{7.5b}$$

where $\not{p}$ and χ are given by (4.6*b*) and (5.16), and letting $\alpha\beta \to \infty$, we transform (7.2) to

$$\Phi'' - \alpha\beta\phi\Phi = -\alpha\beta\Psi. \tag{7.6}$$

Separating Ψ into $\Psi(\phi) - \Psi_n$ and

$$\Psi_n \equiv \Psi(0) = (2\lambda^3)^{-\frac{1}{2}} \lim_{\mu\to\lambda} (\mu^2-\lambda^2)\,\mathsf{U}_n(\mu) \tag{7.7a}$$

$$= (2\lambda^3)^{-\frac{1}{2}}\{2\beta\lambda^2\lambda_* H_n(\lambda) + (m\lambda - \lambda_*^2)\,U_n(\lambda) + m\lambda V_n(\lambda)\}, \tag{7.7b}$$

we obtain the general solution

$$\Phi(\phi) = \pi\kappa\Psi_n\{\mathrm{Gi}\,(\kappa\phi) + C\,\mathrm{Ai}\,(\kappa\phi)\} + \{\Psi(\phi)-\Psi_n\}\,\phi^{-1} + O(1/\alpha\beta), \tag{7.8}$$

where $$\mathrm{Ai}\,(\xi) = \pi^{-1}\int_0^\infty \cos\,(\tfrac{1}{3}t^3 + \xi t)\,dt \tag{7.9a}$$

$$\sim \begin{cases} \tfrac{1}{2}\pi^{-\frac{1}{2}}\xi^{-\frac{1}{4}}\exp\,(-\tfrac{2}{3}\xi^{\frac{3}{2}})\,\{1+O(\xi^{-\frac{3}{2}})\} & (\xi\to\infty) \quad (7.9b)\\ \pi^{-\frac{1}{2}}(-\xi)^{-\frac{1}{4}}\sin\{\tfrac{2}{3}(-\xi)^{\frac{3}{2}}+\tfrac{1}{4}\pi\} + O(\xi^{-\frac{7}{4}}) & (\xi\to-\infty) \quad (7.9c)\end{cases}$$

and $$\mathrm{Gi}\,(\xi) = \pi^{-1}\int_0^\infty \sin\,(\tfrac{1}{3}t^3 + \xi t)\,dt \tag{7.10a}$$

$$\sim \begin{cases} (\pi\xi)^{-1} + O(\xi^{-3}) \quad (\xi\to\infty) & (7.10b)\\ (\pi\xi)^{-1} + \pi^{-\frac{1}{2}}(-\xi)^{-\frac{1}{4}}\cos\{\tfrac{2}{3}(-\xi)^{\frac{3}{2}}+\tfrac{1}{4}\pi\} + O(\xi^{-\frac{7}{4}}) \quad (\xi\to-\infty), & (7.10c)\end{cases}$$

which are plotted in figure 3, are Airy functions (Abramowitz & Stegun 1964, §10.5) and C is a constant. The complementary solution $\mathrm{Bi}\,(\xi)$ is proscribed by (7.3*a*). The particular solution $\{\Psi(\phi)-\Psi_n\}/\phi$, which is regular at $\phi = 0$ $(\mu = \lambda)$, is negligible compared with the Airy solution for $\kappa\phi = O(1)$ but is significant (within the antecedent approximations) for $\kappa\phi \gg 1$. Substituting (7.8) into (7.4*a*) and invoking (7.9*c*), (7.10*c*) and either (7.3*b*) or (7.3*c*), we obtain

$$C = \tan\{(\alpha\beta)^{\frac{1}{2}}\chi_0 \mp \tfrac{1}{4}\pi\}, \tag{7.11}$$

where χ_0 is given by (5.16*b*) and, here and subsequently, the upper/lower signs correspond to odd/even $\mathscr{U}(\mu)$ and, therefore, even/odd $H_n(\mu)$.

Substituting (7.8) and (7.11) into (7.4*a*), substituting the result into (6.8*b*), solving for $\mathscr{V}$ and $\mathscr{P}$, restoring the subscripts j and n and retaining only the dominant terms, we obtain

$$\{\mathscr{U}_j, \mathscr{V}_j\} \sim \mathscr{A}_j\delta[\{\mathsf{U}_n, \mathsf{V}_n\} + \Psi_n\mu_*^{-1}\{\lambda,\mu\}\Upsilon_{jn}] \tag{7.12a}$$

and $$\mathscr{P}_j \sim \mathscr{A}_j\delta(\mu_* V_n + \Psi_n\Pi_{jn}) \quad (\alpha_j\beta_n \to \infty), \tag{7.12b}$$

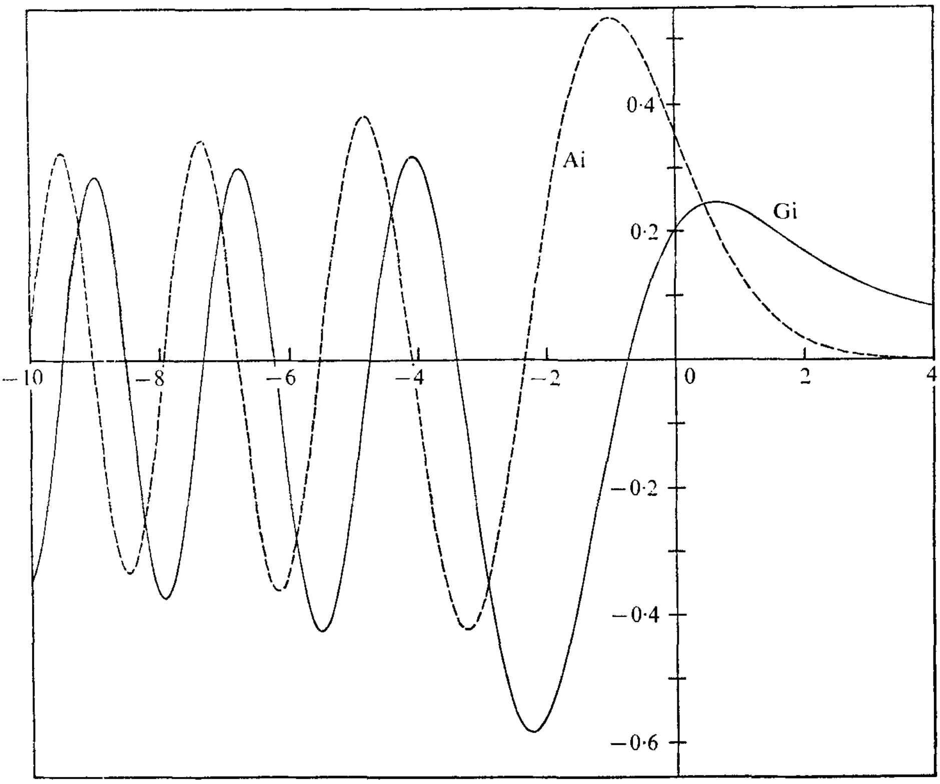

FIGURE 3. The Airy functions Ai and Gi; see (7.9) and (7.10).

where

$$\Upsilon(\phi) = \pi\kappa(-p\phi)^{\frac{1}{4}}\{\mathrm{Gi}(\kappa\phi) - (\pi\kappa\phi)^{-1} + C_{jn}\,\mathrm{Ai}(\kappa\phi)\} \tag{7.13a}$$

$$\sim \pi^{\frac{1}{2}}(\alpha\beta p)^{\frac{1}{4}}\sec\{(\alpha\beta)^{\frac{1}{2}}\chi_0 \mp \tfrac{1}{4}\pi\}\,{\scriptstyle\begin{matrix}\sin\\ \cos\end{matrix}}\{(\alpha\beta)^{\frac{1}{2}}(\chi_0-\chi)\} \quad (\kappa\phi \to -\infty), \tag{7.13b}$$

$$\Pi(\phi) = -\pi\kappa^{-1}(-p\phi)^{-\frac{1}{4}}\{\mathrm{Gi}'(\kappa\phi) + C_{jn}\mathrm{Ai}'(\kappa\phi)\} \tag{7.14a}$$

$$\sim \mp\, \pi^{\frac{1}{2}}(\alpha\beta p)^{-\frac{1}{4}}\sec\{(\alpha\beta)^{\frac{1}{2}}\chi_0 \mp \tfrac{1}{4}\pi\}\,{\scriptstyle\begin{matrix}\cos\\ \sin\end{matrix}}\{(\alpha\beta)^{\frac{1}{2}}(\chi_0-\chi)\} \quad (\kappa\phi \to -\infty), \tag{7.14b}$$

κ, ϕ, C, χ_0 and χ are given by (7.1c), (7.5), (7.11) and (5.16), and the subscripts j and n imply $\alpha = \alpha_j$ and $\beta = \beta_n$. The terms $\{\mathsf{U}_n, \mathsf{V}_n\}$ and $\mu_* V_n$ in (7.12a, b) correspond to (6.14) and represent the solution in $\alpha_j\beta_n(\mu^2-\lambda^2) \gg 1$. The remaining terms represent internal waves in $\mu^2 < \lambda^2$ that are excited by the basic solution, which is represented by Ψ_n. Substituting the latter terms into (6.1a), calculating the corresponding result for Z with the aid of (5.11) and (6.2) and appending the subscript n and the superscript i (for *internal wave*), we obtain

$$\{P_n^{(i)}, Z_n^{(i)}\} = \delta\Psi_n \sum_{j=1}^{\infty} \mathscr{A}_j\{F_j(z), \alpha_j G_j(z)\}\,\Pi_{jn}(\phi) \tag{7.15a}$$

and

$$\{U_n^{(i)}, V_n^{(i)}\} = \delta\Psi_n \mu_*^{-1}\{\lambda, \mu\} \sum_{j=1}^{\infty} \mathscr{A}_j F_j(z)\,\Upsilon_{jn}(\phi) \tag{7.15b}$$

for the internal waves excited by the nth normal mode of (4.3). Adding the solutions of (6.17) and (7.15) yields the first-order perturbation solution of (2.6) and (2.9) with an error that is second order in Γ and $\beta_n \delta$, uniformly with respect to μ.

The eigensolutions of (4.3) for $\beta \to \infty$ may be obtained by setting $\alpha = 1$ and $\mathscr{A} = 0$ in (6.8b) and repeating the preceding analysis. The end results, normalized according to (4.8), are given by (cf. Hughes 1964)

$$H \sim -\beta^{-\frac{1}{12}}(\chi_0/\pi)^{-\frac{1}{2}}(-\not{p}\phi)^{-\frac{1}{4}}\,\mathrm{Ai}'(\beta^{\frac{1}{3}}\phi), \tag{7.16a}$$

$$(\mu_*/\lambda)\,U \sim (\mu_*/\mu)\,V \sim \beta^{\frac{7}{12}}(\chi_0/\pi)^{-\frac{1}{2}}(-\not{p}\phi)^{\frac{1}{4}}\,\mathrm{Ai}(\beta^{\frac{1}{3}}\phi) \tag{7.16b}$$

and

$$\beta \sim \{(l-\tfrac{1}{2})\pi/2\chi_0\}^2 \equiv \beta_l, \tag{7.16c}$$

where l is the number of zeros of H in $|\mu| < \lambda$ (H has no zeros in $\lambda < |\mu| < 1$). Invoking (7.9c) and replacing l by n yields (5.17). The corresponding resonances for (7.15) are given by $\alpha_j \beta_n = \beta_l$, where l is any large positive integer.

The internal-wave displacement and current given by (7.15a) and (7.15b) typically would be somewhat less and much smaller, respectively, than their basic counterparts (roughly 60 % and $\frac{1}{2}$ % if $s = 0{\cdot}002$ and $\beta = 20$) were it not for the possibility of resonance. A suitable global measure for a particular mode is the ratio of the baroclinic and barotropic potential energies, which is given by [cf. left-hand side of (3.17)]

$$\mathscr{E}_n^{(i)} \equiv \int_{-1}^{0} s\,dz \int_{-1}^{1} |Z_n^{(i)}|^2\,d\mu \Big/ \int_{-1}^{1} H_n^2\,d\mu \tag{7.17}$$

within $O(\Gamma, \delta)$. Invoking (4.8), (7.15a), $\mathscr{A}_j = -\langle G_j \rangle$ from (6.7b), the orthonormality of the G_j, (5.12c), and the asymptotic approximation (7.14b), we obtain

$$\mathscr{E}_n^{(i)} = \pi(\delta \Psi_n)^2 \chi_0 \beta_n^{-\frac{1}{2}} \sum_{j=1}^{\infty} \alpha_j^{\frac{1}{2}} \langle G_j \rangle^2 \sec^2\{(\alpha_j \beta_n)\,\chi_0 \mp \tfrac{1}{4}\pi\}, \tag{7.18}$$

which would be of the order of $\beta_n \delta / \langle \mathscr{N} \rangle$ (roughly 10^{-3}) in the absence of resonance. The analysis is not uniformly valid in the spectral neighbourhood of a strong resonance, where it must be modified to allow for either the contribution of the resonant mode to the right-hand side of (7.2) or dissipation (one of my students, Mr Craig Nelson, is carrying out the former modification). The model of a global ocean of uniform depth is too simplified, and our knowledge of stratification in the real oceans is too limited, to say more.

8. Conclusions

Laplace's approximation, which implies the reduction of (2.6) and (2.9) to (4.2) and (4.3), provides a uniformly valid approximation to the barotropic (surface-wave) tidal modes for a global ocean with a relative error of $O(\Gamma, \delta, m)$. It does not describe the baroclinic (internal-wave) tidal modes, which are a consequence of buoyancy and may dominate the vertical displacement below the free surface.

The barotropic and baroclinic modes are essentially independent if $\lambda > 1$, but are coupled by the μ_* component of the Coriolis acceleration (i.e. that component associated with the horizontal component of the Earth's rotation) if $\lambda < 1$. The simplest model that provides a uniformly valid description in the parametric domain of the real oceans ($\Gamma, \delta, \mathscr{m} \ll 1$ and $\lambda < 1 \ll \mathscr{N}$) retains the buoyancy term in the basic equations and poses the μ_* component of the Coriolis acceleration as a perturbation coupling term; it neglects compressibility and the non-buoyant effects of stratification (the Boussinesq approximation), the radial variation of both gravity and the metrical coefficients, and ellipticity, thereby incurring relative errors that are uniformly $O(\Gamma, \delta, \mathscr{m})$, respectively. The resulting reduction of (2.6) yields

$$sZ + P_z = \delta\mu_* V \tag{8.1}$$

and

$$\begin{bmatrix} \lambda & -\mu & -\mu_*\partial_\mu \\ -\mu & \lambda & m/\mu_* \\ \partial_\mu\mu_* & m/\mu_* & -\beta\lambda\,\partial_z s^{-1}\partial_z \end{bmatrix} \begin{bmatrix} U \\ V \\ P \end{bmatrix} = -\beta\delta\lambda\mu_* \begin{bmatrix} 0 \\ Z \\ (V/s)_z \end{bmatrix} \tag{8.2}$$

in place of (4.1) and (4.3). The boundary conditions are given by (2.9).

The barotropic solutions of (8.1), (8.2) and (2.9) are given within $O(\beta\delta)$ by

$$\{P, Z\} = \{1, 1+z\} H_n(\mu), \quad \beta = \beta_n \tag{8.3a, b}$$

and

$$\{U, V\} = \mathscr{U}P, \tag{8.4}$$

where the set $\{H_n;\, \beta_n\}$ is determined by (4.5), together with finiteness conditions at $\mu = \pm 1$, and the operator $\mathscr{U}$ is defined by (4.4). The corresponding baroclinic solutions are given by (8.4) and

$$\{P, Z\} = \{F_j(z), \alpha_j G_j(z)\} H_n(\mu), \quad \beta = \beta_n/\alpha_j, \tag{8.5a, b}$$

where the sets $\{F_j;\, \alpha_j\}$ and $\{G_j;\, \alpha_j\}$ are determined by (5.10) and (5.12), respectively, and are linearly related by (5.11). A qualitatively (quantitatively for $j \gg 1$) adequate approximation for internal-wave resonance is given by (5.19).

Any one of the modes specified by (8.3) and (8.4) may be used as the base for a perturbation solution of (8.1) and (8.2); however, the perturbation is significant within the antecedent approximations only if $\lambda < 1$. The perturbation of a barotropic mode yields (7.15); the resulting complex amplitude of the vertical displacement is

$$Z = (1+z) H_n(\mu) - \delta\Psi_n \sum_{j=1}^{\infty} \alpha_j \langle G_j \rangle G_j(z)\, \Pi_{jn}(\phi), \tag{8.6}$$

where Ψ_n and Π_{jn} are given by (7.7b) and (7.14). The baroclinic component of (8.6) is typically larger than the barotropic component for $z < 0$ and might be very much larger in consequence of resonance.

The implications of these results for tidal motion in the real oceans are limited by our neglect of continental boundaries and submarine topography and by the lack of quantitatively adequate descriptions of stratification (which may exhibit random variations). Moreover, the results are not valid near a strong resonance, and it is possible that a realistic treatment of internal tides requires a stochastic analysis (as in architectural acoustics). Nevertheless, the present results resolve earlier controversy over the 'traditional approximation' and establish the existence of a uniformly valid approximation for $N \gg \sigma$.

I am indebted to W. H. Munk and G. W. Platzman for helpful comments and bibliographical aid. This work was partially supported by the Atmospheric Sciences Section, National Science Foundation, NSF Grant GA-35396X1, and by the Office of Naval Research.

Appendix. Transformation of (2.6)

Multiplying the first two rows of (2.6*a*) through by $\hat{\rho}F$, integrating over $(-1, 0)$ and invoking (2.6*d*) and (6.1*b*) yields

$$\begin{bmatrix} \lambda & -\mu \\ -\mu & \lambda \end{bmatrix} \begin{bmatrix} \mathscr{U} \\ \mathscr{V} \end{bmatrix} = \begin{bmatrix} \mu_* \partial_\mu \\ -m/\mu_* \end{bmatrix} \mathscr{P} - \beta\delta\mu_* \begin{bmatrix} 0 \\ \langle \hat{\rho}FZ \rangle \end{bmatrix}. \tag{A 1}$$

Multiplying (6.2) through by $\hat{\rho}(F' - sF)$, integrating over $(-1, 0)$, invoking the partial integration

$$\langle \hat{\rho}(s^{-1}F' - F) P_z \rangle = \hat{\rho}(s^{-1}F' - F) P \Big|_{-1}^{0} - \langle \{(\hat{\rho}F'/s)' - (\hat{\rho}F)'\} P \rangle \tag{A 2a}$$

$$= -(\hat{\rho}FZ)_0 + \alpha\mathscr{P} + \langle (\hat{\rho}F)' P \rangle, \tag{A 2b}$$

where (A 2*b*) follows from (A 2*a*) with the aid of (2.9*a*), (5.6), (5.5) and (6.1*b*), and substituting

$$\hat{\rho}' = -\hat{\rho}(s + \Gamma) \tag{A 3}$$

from (2.7), we obtain

$$\langle \hat{\rho}(F' - sF) Z \rangle = (\hat{\rho}FZ)_0 - \alpha\mathscr{P} + \Gamma\langle \hat{\rho}FP \rangle + \delta\langle \hat{\rho}(s^{-1}F' - F)(\mu_* V + \lambda W) \rangle. \tag{A 4}$$

Rewriting (2.6*b*) in the form (5.1*c*), multiplying through by $\hat{\rho}F$ and invoking the partial integration

$$\langle \hat{\rho}FZ_z \rangle = \hat{\rho}FZ \Big|_{-1}^{0} - \langle (\hat{\rho}F)'Z \rangle \tag{A 5a}$$

$$= (\hat{\rho}FZ)_0 - \langle \{\hat{\rho}(F' - sF) - \Gamma\hat{\rho}F\} Z \rangle, \tag{A 5b}$$

where (*A* 5*b*) follows from (A 5*a*) with the aid of (2.9*b*) and (*A* 3), we obtain

$$(\mu_* \mathscr{U})_\mu + (m/\mu_*)\mathscr{V} + \beta\lambda\{(\rho FZ)_0 - \langle \hat{\rho}(F' - sF) Z \rangle + \Gamma\langle \hat{\rho}FP \rangle\} = 0. \tag{A 6}$$

Combining (A 4) and (A 6) yields

$$(\mu_* \mathscr{U})_\mu + (m/\mu_*)\mathscr{V} + \alpha\beta\lambda\mathscr{P} = \beta\delta\lambda\langle \hat{\rho}(s^{-1}F' - F)(\mu_* V + \lambda W) \rangle. \tag{A 7}$$

Combining (A 1) and (A 7) yields (6.3).

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